

Advanced Counting Techniques

Chapter 8

With Question/Answer Animations

Chapter Summary

- Applications of **Recurrence** Relations
- Solving **Linear** Recurrence Relations
 - **Homogeneous** Recurrence Relations
 - **Nonhomogeneous** Recurrence Relations
- **Divide-and-Conquer** Algorithms and Recurrence Relations
- Generating Functions
- **Inclusion-Exclusion**
- Applications of Inclusion-Exclusion

Recurrence 递推关系
Linear 线性
Homogeneous 齐次
Nonhomogeneous 非齐次
Inclusion-Exclusion 包含互斥
Divide-and-Conquer 分治

Applications of Recurrence Relations

Section 8.1

Section Summary

- Applications of Recurrence Relations
 - **Fibonacci** Numbers
 - The Tower of **Hanoi**
 - Counting Problems

Fibonacci 斐波那契
Hanoi 汉诺

Recurrence Relations

(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence *specify* the terms that *precede* the first term where the recurrence relation takes effect.

Initial 初始
Precede 之前
Specify 指明

Rabbits and the Fibonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not **breed** until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century. Breed 繁殖

Rabbits and the Fibonacci Numbers (cont.)

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
	 	4	1	2	3
 	  	5	2	3	5
  	   	6	3	5	8
	 				

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers (cont.)

Solution: Let f_n be the the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1} , and the number of newborn pairs, which equals f_{n-2} , because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$. The number of pairs of rabbits on the island after n months is given by the n th Fibonacci number.

The Tower of Hanoi

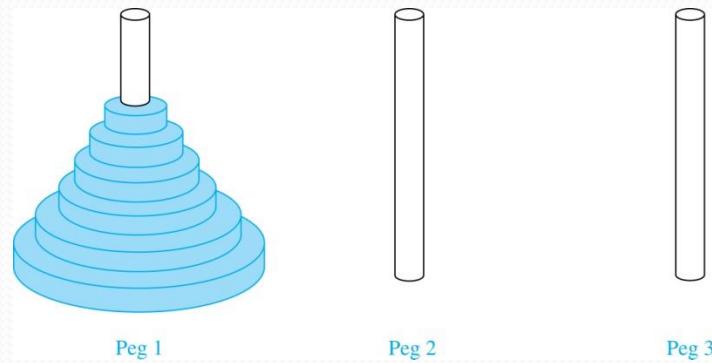
In the late nineteenth century, the French mathematician Édouard Lucas invented a **puzzle** consisting of three **pegs** on a **board** with **disks** of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

Puzzle 难题
Board 木板
Peg 钉子
Disk 圆盘

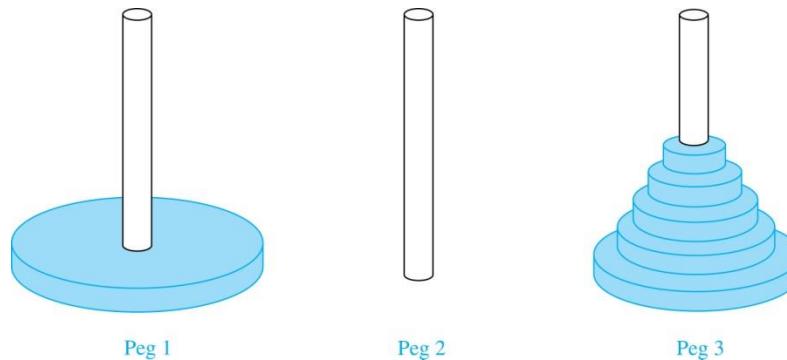
The Tower of Hanoi (*continued*)



The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi (*continued*)

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n - 1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi (*continued*)

- We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$\begin{aligned}H_n &= 2H_{n-1} + 1 \\&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\&\vdots \\&= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \\&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \quad \text{because } H_1 = 1 \\&= 2^n - 1 \quad \text{using the formula for the sum of the terms of a geometric series}\end{aligned}$$

Geometric series 等比数列
Conjecture 推测

- There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
- Using this formula for the 64 gold disks of the myth,
 $2^{64} - 1 = 18,446,744,073,709,551,615$ days are needed to solve the puzzle, which is more than 500 billion years.
- Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known **unsettled conjecture** for the the minimum number of moves needed to solve this puzzle. (see Exercises 38-45)

Counting Bit Strings

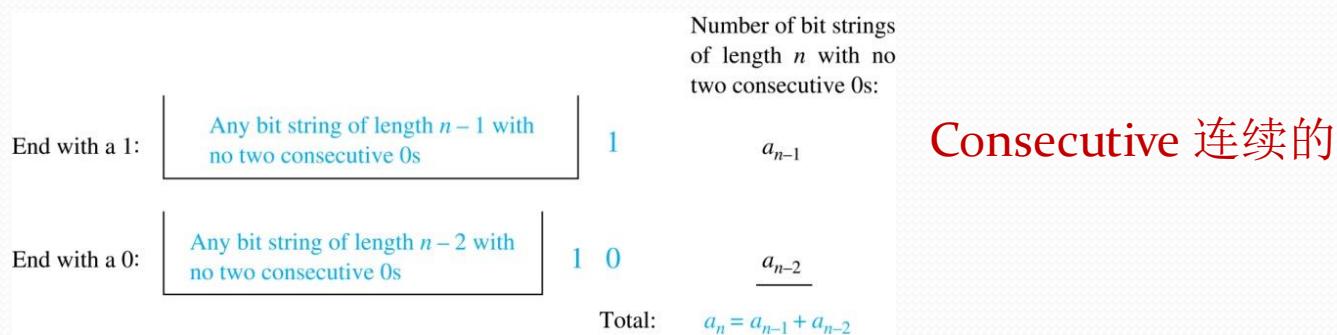
Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the **number of bit strings ending with a 0 plus the number of such bit strings ending with a 1**.

Now assume that $n \geq 3$.

- The bit strings of length n ending with 1 without two consecutive 0s/ are the bit strings of length $n-1$ with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.
- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length $n-2$ with no two consecutive 0s with 10 at the end. Hence, there are a_{n-2} such bit strings.

We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.



Bit Strings (*continued*)

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5 , we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

$f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3 \dots$

Counting the Ways to Parenthesize a Product

Example: Find a recurrence relation for C_n , the number of ways to **parenthesize** the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \dots \cdot x_n$, to specify the order of multiplication.

For example, $C_3 = 5$, since all the possible ways to parenthesize 4 numbers are

$$((x_0 \cdot x_1) \cdot x_2) \cdot x_3, \quad (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3, \quad (x_0 \cdot x_1) \cdot (x_2 \cdot x_3), \quad x_0 \cdot ((x_1 \cdot x_2) \cdot x_3), \quad x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$$

Solution: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \dots \cdot x_n$, one “ \cdot ” operator remains outside all parentheses. This final operator appears between two of the $n + 1$ numbers, say x_k and x_{k+1} . Since there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \dots \cdot x_k$ and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \dots \cdot x_n$, we have

$$\begin{aligned} C_n &= C_0 C_{n-1} + C_1 C_{n-2} + \dots + C_{n-2} C_1 + C_{n-1} C_0 \\ &= \sum_{k=0}^{n-1} C_k C_{n-k-1} \end{aligned}$$

Parenthesize 括号

The initial conditions are $C_0 = 1$ and $C_1 = 1$.

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**. This recurrence relation can be solved using the method of generating functions; see Exercise 41 in Section 8.4.

Catalan Numbers: <https://baike.baidu.com/item/卡特兰数/6125746>

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with **Constant Coefficients**.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Constant 常量
Coefficients 系数

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$, where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$

- it is ***linear*** because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n .
- it is ***homogeneous*** because no terms occur that are not multiples of the a_i s. Each coefficient is a constant.
- the ***degree*** is k because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1, a_1 = C_2, \dots, a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the *characteristic equation*:
$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$
- **The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.**
- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Characteristic 特征

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 . Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \dots$, where α and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution: The characteristic equation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$, for some constants α_1 and α_2 .

To find the constants α_1 and α_2 , note that

$$a_0 = 2 = \alpha_1 + \alpha_2 \text{ and } a_1 = 7 = \alpha_1 2 + \alpha_2 (-1).$$

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an **explicit** formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Explicit 显式的

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2 .

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1 .$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has one repeated root r_0 . Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0, 1, 2, \dots$, where α and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.

The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2 , note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.

Hence,

$$a_n = 3^n + n3^n.$$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let c_1, c_2, \dots, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1r^{k-1} - \dots - c_k = 0$$

has k distinct roots r_1, r_2, \dots, r_k . Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^m + \dots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \dots$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let c_1, c_2, \dots, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \dots - c_k = 0$$

has t distinct roots r_1, r_2, \dots, r_t with **multiplicities** m_1, m_2, \dots, m_t , respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\ & + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n \end{aligned}$$

for $n = 0, 1, 2, \dots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Multiplicities 重数

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n),$$

where c_1, c_2, \dots, c_k are real numbers, and $F(n)$ is a function not **identically** zero depending only on n .

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

Identically 恒定地

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (cont.)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$$a_n = a_{n-1} + 2^n,$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1,$$

$$a_n = 3a_{n-1} + n3^n,$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

where the following are the associated linear homogeneous recurrence relations, respectively:

$$a_n = a_{n-1},$$

$$a_n = a_{n-1} + a_{n-2},$$

$$a_n = 3a_{n-1},$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$